

# ON PROPERTIES OF COMPACTA THAT DO NOT REFLECT IN SMALL CONTINUOUS IMAGES

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**ABSTRACT.** Assuming that there is a stationary set of ordinals of countable cofinality in  $\omega_2$  that does not reflect, we prove that there exists a compact space which is not Corson compact and whose all continuous images of weight  $\leq \omega_1$  are Eberlein compacta. This yields an example of a Banach space of density  $\omega_2$  which is not weakly compactly generated but all its subspaces of density  $\leq \omega_1$  are weakly compactly generated.

We also prove that under Martin's axiom countable functional tightness does not reflect in small continuous images of compacta.

## 1. INTRODUCTION

There is a significant amount of research related to properties of structures that reflect in substructures of smaller cardinality, see e.g. Bagaria, Magidor, Sakai [3], Koszmider [13, 14], Fuchino and Rinot [9], Tall [23]. Reflection phenomena in topology are usually studied according to the following pattern:

**Problem 1.1.** *Does a topological space  $X$  has a property  $(P)$  provided all its subspaces of small cardinality have property  $(P)$ ?*

Tall [23] surveys results and problems of this type. Recently Tkachuk [24] and Tkachuk and Tkachenko [25] have investigated which topological properties reflect in small continuous images which, in particular, amounts to asking the following kind of questions.

**Problem 1.2.** *Does a topological space  $X$  has property  $(P)$  provided every continuous image of  $X$  of weight  $\leq \omega_1$  has property  $(P)$ ?*

Eberlein compacta and Corson compacta are two well-studied classes of compact spaces related to functional analysis, see the next section. Answering two questions of type 1.2 posed in [25], we show in this note that it is relatively consistent that neither Eberlein compactness nor Corson compactness reflects in continuous images of weight  $\leq \omega_1$ . In fact, assuming that there is a stationary set  $S \subseteq \omega_2$  of ordinals of countable cofinality such that

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$S \cap \alpha$  is stationary in no  $\alpha < \omega_2$ , we construct a compact space  $K$  of weight  $\omega_2$  which simultaneously answers in the negative both the questions:  $K$  is not Corson compact while all its images of weight at most  $\omega_1$  are Eberlein compacta that can be embedded into a Hilbert space. In addition, our space  $K$  gives partial negative answers to problems posed by Jardón and Tkachuk ([10], Questions 4.13-15) on the reflection of type 1.1 for Corson compacta and related classes. The construction of the space is given in section 3 and uses the familiar idea of a ladder system associated to a given set  $S \subseteq \omega_2$ ; see, for instance, Pol [22] and Ciesielski and Pol [6] where a construction of this type was used to solve some problems on the structure of  $C(K)$  spaces. The existence of a stationary set in  $\omega_2$  with the above-mentioned properties is known to have an impact on other topological problems, see Fleissner [8], 3.10.

In the framework of Banach space theory our result shows that it is relatively consistent that there exists a Banach space  $X$  of density  $\omega_2$  such that  $X$  is not weakly compactly generated while every subspace  $Y \subseteq X$  of density  $\leq \omega_1$  is weakly compactly generated, see section 4.

In the final section of this note we give a partial negative answer to another problem from [25]. We show, assuming a weak version of Martin's axiom, that countable functional tightness does not reflect in small continuous images of compact spaces.

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## 2. PRELIMINARIES

All the topological spaces we consider here are assumed to be Hausdorff. Given a topological space  $X$ ,  $w(X)$  denotes its topological weight, i.e. the minimal size of a base in  $X$ . Recall that a family  $\mathcal{V}$  of nonempty open subsets of  $X$  is a  $\pi$ -base if every nonempty open set in  $X$  contains some  $V \in \mathcal{V}$ .

Our examples will be constructed from some Boolean algebras. If  $\mathfrak{A}$  is a Boolean algebra then we write  $\text{ult}(\mathfrak{A})$  for its Stone space (of all ultrafilters on  $\mathfrak{A}$ ). We write  $\hat{a} = \{x \in \text{ult}(\mathfrak{A}) : a \in x\}$  for  $a \in \mathfrak{A}$ . Recall that sets  $\hat{a}$  form a base for the topology on  $\text{ult}(\mathfrak{A})$ .

We shall use the following result which is a very particular case of the Mardešić factorization theorem [18]. We enclose the sketch of a direct argument.

**Theorem 2.1.** *Let  $\mathfrak{A}$  be any Boolean algebra. If  $L$  is a continuous image of  $\text{ult}(\mathfrak{A})$  and  $w(L) \leq \omega_1$  then there is a subalgebra  $\mathfrak{B} \subseteq \mathfrak{A}$  such that  $|\mathfrak{B}| \leq \omega_1$  and  $L$  is a continuous image of  $\text{ult}(\mathfrak{B})$ .*

*Proof.* Note that  $L$  has a base  $\mathcal{U}$  of cardinality  $\leq \omega_1$  such that every  $U \in \mathcal{U}$  is  $F_\sigma$ . Let  $f : \text{ult}(\mathfrak{A}) \rightarrow L$  be a continuous surjection. For every  $U \in \mathcal{U}$ , the set  $f^{-1}(U)$  is of type  $F_\sigma$  so it can be written as a union of countably many sets of the form  $\hat{a}$ ,  $a \in \mathfrak{A}$ .

It follows that there is  $\mathfrak{B} \subseteq \mathfrak{A}$  of size at most  $\omega_1$  such that, writing  $\pi : \text{ult}(\mathfrak{A}) \rightarrow \text{ult}(\mathfrak{B})$  for the natural projection, we have  $f(x) = f(y)$  whenever  $x, y \in \text{ult}(\mathfrak{A})$  and  $\pi(x) = \pi(y)$ . Hence we can write  $f = f' \circ \pi$ , where  $f' : \text{ult}(\mathfrak{B}) \rightarrow L$ . It follows that  $f'$  is continuous and the proof is complete.  $\square$

A compact space  $K$  is said to be *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space; equivalently, by the classical Amir-Lindenstrauss theorem,  $K$  is Eberlein compact if it can be embedded into

$$c_0(\kappa) = \{x \in \mathbb{R}^\kappa : \{\alpha : |x_\alpha| \geq \varepsilon\} \text{ is finite for every } \varepsilon > 0\},$$

for some  $\kappa$ . Here  $c_0(\kappa)$  is equipped with the topology inherited from  $\mathbb{R}^\kappa$  (this topology agrees on bounded sets with the weak topology of the Banach space  $c_0(\kappa)$ ).

In particular, if  $n \in \omega$  then every compact subset of

$$\sigma_n(2^\kappa) = \{x \in 2^\kappa : |\{\alpha : |x_\alpha| \neq 0\}| \leq n\},$$

is Eberlein compact. In fact it is uniform Eberlein compact in the sense that it can be embedded as a weakly compact subspace of a Hilbert space (note that  $\sigma_n(2^\kappa)$  is a bounded subset of  $l_2(\kappa)$ ).

A compact space  $K$  is said to be *Corson compact* if there is  $\kappa$  such that  $K$  is homeomorphic to a subset of the  $\Sigma$ -product of real lines

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$

Since  $c_0(\kappa) \subseteq \Sigma(\mathbb{R}^\kappa)$ , the class of Corson compacta contains (properly) the class of Eberlein compacta. Negreponitis [19] and Kalenda [11] offer extensive surveys on Eberlein and Corson compacta and related classes. We only recall here that both uniform Eberlein compacta and Corson compacta are stable under continuous images, see e.g. [19], 6.26 and [11], p. 2.

A family  $\mathcal{F}$  in a Boolean algebra is said to be *centred* if  $a_1 \cap a_2 \cap \dots \cap a_k \neq 0$  for every natural number  $k$  and every  $a_i \in \mathcal{F}$ . We shall use the following standard fact.

**Lemma 2.2.** *For a Boolean algebra  $\mathfrak{A}$  the following are equivalent:*

- (i)  *$\text{ult}(\mathfrak{A})$  is Corson compact;*
- (ii) *there is a family  $\mathcal{G} \subseteq \mathfrak{A}$  generating  $\mathfrak{A}$  and such that every centred subfamily of  $\mathcal{G}$  is countable.*

*Proof.* (i)  $\rightarrow$  (ii). Since  $\text{ult}(\mathfrak{A})$  is Corson compact and zerodimensional,  $\text{ult}(\mathfrak{A})$  is homeomorphic to a compact space  $K$  contained in  $\Sigma(2^\kappa)$  for some  $\kappa$ . The algebra of clopen subsets of  $K$  is generated by the family  $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ , where  $C_\alpha = \{x \in K : x_\alpha = 1\}$ . Every centred subfamily of  $\mathcal{C}$  is countable by the definition of  $\Sigma(2^\kappa)$ .

(ii)  $\rightarrow$  (i). Take  $f : \text{ult}(\mathfrak{A}) \rightarrow 2^\mathcal{G}$ , where  $f(x)(G) = 1$  if  $G \in x$  and  $= 0$  otherwise. Then  $f$  is continuous, and  $f[\text{ult}(\mathfrak{A})] \subseteq \Sigma(2^\mathcal{G})$  since every ultrafilter on  $\mathfrak{A}$  contains at most countably many generators from  $\mathcal{G}$ . Moreover,  $f$  is injective since  $\mathcal{G}$  generates  $\mathfrak{A}$ .  $\square$

### 3. ON EBERLEIN AND CORSON COMPACTA

Let  $\gamma$  be a limit ordinal. A set  $F \subseteq \gamma$  is said to be *closed* if it is closed in the interval topology defined on ordinals smaller than  $\gamma$ . Such a set  $F$  is unbounded in  $\gamma$  if for every  $\beta < \gamma$  there is  $\alpha \in F$  such that  $\beta < \alpha$ . A set  $S \subseteq \gamma$  is *stationary* if  $S \cap F \neq \emptyset$  for every closed and unbounded  $F \subseteq \gamma$ .

It is not difficult to check that the set  $S_\omega = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega\}$  is stationary in  $\omega_2$ . However, such a set reflects in the sense that, for instance,  $S_\omega \cap \omega_1$  is stationary in  $\omega_1$ . We shall work assuming the following.

**Axiom 3.1.** There is a stationary set  $S \subseteq \omega_2$  such that

- (a)  $\text{cf}(\alpha) = \omega$  for every  $\alpha \in S$ ;
- (b)  $S \cap \beta$  is not stationary in  $\beta$  for every  $\beta < \omega_2$  with  $\text{cf}(\beta) = \omega_1$ .

Note that in 3.1(b) we can say that  $S \cap \beta$  is not stationary in  $\beta$  for every limit  $\beta < \omega_2$  because if  $\text{cf}(\beta) = \omega$  then  $\beta$  is a limit of a sequence of successor ordinals.

Basic information on 3.1 can be found in Jech [17]; recall that 3.1 follows from Jensen's principle  $\square_{\omega_1}$  ([17], Lemma 23.6) and hence it holds in the constructible universe ([17], Theorem 27.1). In fact one cannot deny 3.1 and prove the consistency of the statement *every stationary set  $S \subseteq \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega\}$  reflects at some  $\gamma < \omega_2$*  without assuming the existence of large cardinals, see Magidor [16] and [17], page 697.

**Construction 3.2.** Throughout this section we consider the space  $K = \text{ult}(\mathfrak{A})$ , where the Boolean algebra  $\mathfrak{A}$  is defined as follows.

Fix a set  $S \subseteq \omega_2$  as in 3.1. For every  $\alpha \in S$  we pick an increasing sequence  $(p_n(\alpha))_{n < \omega}$  of ordinals such that  $p_n(\alpha) \rightarrow \alpha$ . Put

$$A_\alpha = \{p_n(\alpha) : n < \omega\}, \quad \text{and} \quad X = \bigcup_{\alpha \in S} A_\alpha.$$

Finally, let  $\mathfrak{A}$  be the algebra of subsets of  $X$  generated by finite subsets of  $X$  together with the family  $\{A_\alpha : \alpha \in S\}$ .

We shall prove that  $K = \text{ult}(\mathfrak{A})$  is not Corson compact because  $S$  is stationary in  $\omega_2$  while the absence of stationary reflection for  $S$  implies that  $\text{ult}(\mathfrak{B})$  is Eberlein compact for every small subalgebra  $\mathfrak{B}$  of  $\mathfrak{A}$ .

**Lemma 3.3.** *If  $\mathfrak{A}$  is the algebra defined in 3.2 then the space  $\text{ult}(\mathfrak{A})$  is not Corson compact*

*Proof.* Suppose that there is a family  $\mathcal{G} \subseteq \mathfrak{A}$  as in Lemma 2.2(ii). Note that every  $A \in \mathfrak{A}$  is either countable or co-countable in  $X$ . The family  $\mathcal{G}_0 = \{G \in \mathcal{G} : |X \setminus G| \leq \omega\}$  is centred so it is at most countable. Hence, replacing every  $G \in \mathcal{G}_0$  by its complement, we may assume that every  $G \in \mathcal{G}$  is countable.

Let  $\mathcal{G}_1 = \{G \in \mathcal{G} : |G| = \omega\}$ . Note that every  $G \in \mathcal{G}_1$  is, modulo a finite set, a finite union of sets  $A_\alpha$ .

For every  $\alpha \in S$  there must be  $G_\alpha \in \mathcal{G}_1$  such that  $|A_\alpha \cap G_\alpha| = \omega$ . Indeed, otherwise  $A_\alpha$  would be almost disjoint from every  $G \in \mathcal{G}_1$  so would not be in the algebra generated by  $\mathcal{G}$ . Note that the function  $\alpha \rightarrow G_\alpha$  is finite-to-one.

It follows that for every  $\alpha \in S$  there is  $\varphi(\alpha) < \alpha$  such that  $\varphi(\alpha) \in G_\alpha$ . By the pressing-down lemma, there is  $\xi$  such that the set  $\{\alpha \in S : \varphi(\alpha) = \xi\}$  is stationary. It follows that  $\{G \in \mathcal{G}_1 : \xi \in G\}$  is of cardinality  $\omega_2$ , and this is a contradiction.  $\square$

The second part of the argument is based on the following auxiliary result which is stated in a slightly stronger form suitable for inductive argument.

**Lemma 3.4.** *For every  $\beta, \gamma$  such that  $\beta < \gamma < \omega_2$  there is a family*

$$\mathcal{B}(\beta, \gamma) = \{B_\alpha : \alpha \in S \cap (\beta, \gamma)\},$$

*such that*

- (i)  $B_\alpha \subseteq A_\alpha \setminus \beta$  and  $|A_\alpha \setminus B_\alpha| < \omega$  for every  $\alpha \in S \cap (\beta, \gamma)$ ;
- (ii)  $B_\alpha \cap B_{\alpha'} = \emptyset$  whenever  $\alpha, \alpha' \in S \cap (\beta, \gamma)$  and  $\alpha \neq \alpha'$ .

*Proof.* We prove the assertion by induction on  $\gamma$ .

For the successor step  $\gamma \rightarrow \gamma + 1$  there is nothing to prove in case  $\gamma \notin S$ . Suppose  $\gamma \in S$  and take  $\mathcal{B}(\beta, \gamma)$  satisfying (i) and (ii). Then  $B_\alpha \cap A_\gamma$  is finite for every  $\alpha < \gamma$ ,  $\alpha \in S$  and therefore

$$\{B_\alpha \setminus A_\gamma : \alpha \in (\beta, \gamma) \cap S\} \cup \{A_\gamma \setminus \beta\},$$

is the required family for the interval  $(\beta, \gamma)$ .

Suppose that  $\gamma$  is a limit ordinal. Then  $S \cap \gamma$  is not stationary in  $\gamma$  so there is a closed unbounded set  $C \subseteq \gamma$  such that  $C \cap S = \emptyset$ . In other words,  $S \cap \gamma$  is contained in a set  $\gamma \setminus C$  which is open and hence is a union of disjoint subintervals.

Fix  $\beta < \gamma$ . If  $\xi, \eta \in C$ ,  $\beta < \xi < \eta$  and  $(\xi, \eta) \cap C = \emptyset$  then we can apply the inductive assumption to  $S \cap (\xi, \eta)$  and get the required family  $\mathcal{B}(\xi, \eta)$ . The union of families obtained in this way is clearly the family that satisfies (i) and (ii).  $\square$

**Lemma 3.5.** *For every algebra  $\mathfrak{B} \subseteq \mathfrak{A}$ , where  $\mathfrak{A}$  is as in 3.2, if  $|\mathfrak{B}| \leq \omega_1$  then the space  $\text{ult}(\mathfrak{B})$  is uniform Eberlein compact.*

*Proof.* Let  $\gamma < \omega_2$  and let  $\mathfrak{B}_\gamma$  be a subalgebra of  $\mathfrak{A}$  generated by all finite sets in  $X$  and the family  $\{A_\alpha : \alpha < \gamma\}$ . It follows directly from Lemma 3.4 that  $\mathfrak{B}_\gamma$  has a generating family  $\mathcal{G}$  such that there are no three different elements in  $\mathcal{G}$  having nonempty intersection. Then the space  $\text{ult}(\mathfrak{B}_\gamma)$  can be embedded into  $\sigma_2(2^\gamma)$  (as in Lemma 2.2) so it is uniform Eberlein compact.

Now every subalgebra  $\mathfrak{A} \subseteq \mathfrak{A}$  of size  $\leq \omega_1$  is included in  $\mathfrak{B}_\gamma$  for some  $\gamma < \omega_2$ . Hence  $\text{ult}(\mathfrak{B})$  is a continuous image of  $\text{ult}(\mathfrak{B}_\gamma)$  and thus it is uniform Eberlein compact as well.  $\square$

The following answers simultaneously, subject to our set-theoretic assumption, Questions 4 and 5 in [25].

**Theorem 3.6.** *Assume 3.1. There is a scattered compact space  $K$  with the third derivative empty such that*

- (i)  $K$  is not Corson compact (in fact it is not  $\omega_2$ -Corson compact in the sense of [11]);
- (ii) If  $L$  is a continuous image of  $K$  and  $w(L) \leq \omega_1$  then  $L$  is uniform Eberlein compact.

*Proof.* We take  $K = \text{ult}(\mathfrak{A})$ , where  $\mathfrak{A}$  is the algebra defined above in 3.2. Since  $K$  is a Stone space of an algebra generated by an almost disjoint family, it is clear that  $K^{(3)} = \emptyset$ . Indeed, every ultrafilter  $x \in \text{ult}(\mathfrak{A})$  is either principal or there is a unique  $\alpha$  such that  $A_\alpha \in x$  or else  $x \in K^{(2)}$  is the unique ultrafilter containing all  $X \setminus A_\alpha$ .

Then  $K$  is not Corson compact by Lemma 3.3. If  $L$  is a continuous images of  $K$  and  $w(L) \leq \omega_1$  then  $L$  is uniform Eberlein compact by Theorem 2.1, Lemma 3.5 and the fact that uniform Eberlein compacta are stable under continuous images.  $\square$

*Remark 3.7.* In connection with Theorem 3.6, it is worth remarking that since the space  $K$  used in its proof is scattered, every continuous image of  $K$  is zerodimensional so in fact the use of 2.1 is not essential. Moreover, the following results are closely related to the final conclusion on continuous images of  $K$  having small weight.

- (1) Alster [1] proved that every scattered Corson compact space is Eberlein compact.
- (2) Bell and Marciszewski [4] proved that a scattered Eberlein compact space of height at most  $\omega + 1$  is uniform Eberlein compact.

As we mentioned in the introduction, the space  $K$  from Theorem 3.6 settles in the negative some reflection problems of type 1.1.

**Lemma 3.8.** *Let  $(P)$  be a property of compact space that is stable under taking closed subspaces. If  $K$  is a compact space and all continuous images of weight  $\leq \omega_1$  have property  $(P)$  then all closed subsets  $L$  of  $K$  of cardinality  $\leq \omega_1$  have property  $(P)$ .*

*Proof.* Take a closed subspace  $L \subseteq K$  with  $|L| \leq \omega_1$ . Then there is a family  $\mathcal{F}$  of continuous functions  $K \rightarrow [0, 1]$  such that  $|\mathcal{F}| \leq \omega_1$  and  $\mathcal{F}$  separates the points of  $L$ . Let  $g : K \rightarrow [0, 1]^\mathcal{F}$  be the diagonal mapping, i.e.  $g(x)(f) = f(x)$  for  $f \in \mathcal{F}$ . Then  $\tilde{K} = g[K] \subseteq [0, 1]^\mathcal{F}$  so  $w(\tilde{K}) \leq |\mathcal{F}| \leq \omega_1$  and hence  $\tilde{K}$  has property (P). It follows that  $\tilde{L} = g[L] \subseteq \tilde{K}$  also has property (P), and  $\tilde{L}$  is homeomorphic to  $L$ .  $\square$

**Corollary 3.9.** *Assume 3.1 and take the space  $K$  as in Theorem 3.6. Then  $K$  is not Corson compact while for every  $Y \subseteq K$ , if  $|Y| \leq \omega_1$  then  $\overline{Y}$  is uniform Eberlein compact.*

*Proof.* Recall that  $X$  was defined as the union of all the sets  $A_\alpha$ ,  $\alpha \in S$ . If  $Y \subseteq X$  and  $|Y| \leq \omega_1$  then  $Y \subseteq X \cap \gamma$  for some  $\gamma < \omega_2$  and this easily implies that  $|\overline{Y}| \leq \omega_1$ . If  $Y \subseteq K^{(1)}$  then  $\overline{Y} \subseteq Y \cup \{\infty\}$ , where  $\infty$  is the only point in  $K^{(2)}$ .

We conclude that  $|\overline{Y}| \leq \omega_1$  for every  $Y \subseteq K$  with  $|Y| \leq \omega_1$  and the assertion follows from Lemma 3.8.  $\square$



The corollary above gives partial negative answers to problems posed by Jardón and Tkachuk ([10], Questions 4.13-15) if we assume the continuum hypothesis together with 3.1, so for instance if we are in the constructible universe.

#### 4. ON WCG BANACH SPACES

A Banach space  $\mathfrak{X}$  is said to be *weakly compactly generated* (WCG) if there is a weakly compact set  $C \subseteq \mathfrak{X}$  such that  $\mathfrak{X} = \overline{\text{span}}(C)$ , see Negrepolis [19] for a survey on WCG Banach spaces and further references.

Given a compact space  $K$ ,  $C(K)$  denotes the Banach space of real-valued continuous functions on  $K$  (equipped with the usual supremum norm). By a classical result due to Amir and Lindenstrauss, there is a natural duality between the class of WCG Banach spaces and the class of Eberlein compacta. In particular, the following holds (see e.g. [19], Theorem 6.9).

**Theorem 4.1.** *Given a compact space  $K$ , the Banach space  $C(K)$  is WCG if and only if  $K$  is Eberlein compact.*

Let us recall that the class of WCG Banach spaces is not stable under taking subspaces. We shall use the following fact which is a very particular case of a result due to Fabian [7].

**Theorem 4.2** (Fabian). *Let  $K$  be a compact scattered space. If  $\mathfrak{Z}$  is a WCG subspace of  $C(K)$  then every subspace  $\mathfrak{Y}$  of  $\mathfrak{Z}$  is WCG too.*

In the spirit of reflection problems considered above, one can wonder if, for a Banach space  $\mathfrak{X}$ , the property of being WCG reflects in subspaces of small density. The following reformulation of Theorem 3.6 gives a partial negative answer to such a question.

**Theorem 4.3.** *Under 3.1 there exists a Banach space  $\mathfrak{X}$  of density  $\omega_2$  such that*

- (i)  $\mathfrak{X}$  is not weakly compactly generated;
- (ii) every subspace  $\mathfrak{Y} \subseteq \mathfrak{X}$  of density  $\leq \omega_1$  is weakly compactly generated.

*Proof.* We take the space  $K$  as in Theorem 3.6 and consider  $\mathfrak{X} = C(K)$ . Then  $\mathfrak{X}$  is not WCG by Theorem 4.1, since  $K$  is not Eberlein compact. As the weight of  $K$  is  $\omega_2$ , the space  $\mathfrak{X}$  is of density  $\omega_2$ .

Take any subspace  $\mathfrak{Y} \subseteq \mathfrak{X}$  of density  $\leq \omega_1$ . Then there is a family  $\mathcal{F} \subseteq Y$  of size  $\leq \omega_1$  which is dense in  $\mathfrak{Y}$ . Let  $\theta : K \rightarrow \mathbb{R}^{\mathcal{F}}$  be the diagonal mapping, that is  $\theta(x)(f) = f(x)$  for  $x \in K$  and  $f \in \mathcal{F}$ . Then  $L = \theta[K]$  is a continuous image of  $K$  of weight  $\leq \omega_1$  so  $L$  is Eberlein compact by Theorem 3.6(ii) and the Banach space  $C(L)$  is WCG by Theorem 4.1.

The space  $\mathfrak{Z} = \{h \circ \theta : h \in C(L)\}$  is an isometric copy of  $C(L)$  so  $\mathfrak{Z}$  is WCG too. Now  $\mathfrak{Y}$  is clearly a subspace of  $\mathfrak{Z}$  so  $\mathfrak{Y}$  is WCG by Theorem 4.2, and the proof is complete.  $\square$

## 5. ON COUNTABLE FUNCTIONAL TIGHTNESS

**Definition 5.1.** For a topological space  $X$  and a cardinal number  $\kappa$  we write  $t_0(X) \leq \kappa$  if every function  $f : X \rightarrow \mathbb{R}$  is continuous provided  $f|_Y : Y \rightarrow \mathbb{R}$  is continuous for every subspace  $Y \subseteq X$  with  $|Y| \leq \kappa$ . The corresponding cardinal number  $t_0(X)$  is called the *functional tightness* of the space  $X$ .

Recall that  $t(X)$ , the tightness of a space  $X$  is defined so that for every  $A \subseteq X$  and every  $x \in \overline{A}$  there is  $B \subseteq A$  such that  $|B| \leq t(X)$  and  $x \in \overline{B}$ . The following fact can be found in [2].

**Lemma 5.2.** *The functional tightness  $t_0(X)$  does not exceed the density of  $X$  for every space  $X$ . Moreover,  $t_0(X) \leq t(X)$ .*

Tkachuk (see Theorem 2.11 in [24]) proved that if  $K$  is a compact space of uncountable tightness then  $K$  has a continuous image  $L$  of uncountable tightness with  $w(L) = \omega_1$ .

Recall that  $t_0(2^\kappa) = \omega$  if and only if there are no measurable cardinals  $\leq \kappa$ , see Uspenskii [26], cf. [20]. Using this theorem it is noted in [25] that if there are measurable cardinals then the countable functional tightness does not reflect in small continuous images of compacta.

Let  $\lambda$  be the Lebesgue measure on  $[0, 1]$ . Write  $\mathcal{N}$  for the ideal of  $\lambda$ -null sets. Recall that the assertion  $\text{cov}(\mathcal{N}) > \omega_1$  means that  $[0, 1]$  cannot be covered by  $\omega_1$ -many sets from  $\mathcal{N}$ .

We shall work in the measure algebra  $\mathfrak{A}$  of the Lebesgue measure on  $[0, 1]$ ; the corresponding measure on  $\mathfrak{A}$  is still denoted by  $\lambda$ . The following is an immediate consequence of a result due to Kamburelis [12], Lemma 3.1; see also [5], Theorem 4.4.

**Theorem 5.3.** *If  $\text{cov}(\mathcal{N}) > \omega_1$  then every continuous image of  $\text{ult}(\mathfrak{A})$  of  $\pi$ -weight  $\leq \omega_1$  is separable.*

We can now give a (partial) negative solution to Question 4.3 from [25].

**Theorem 5.4.** *Assuming  $\text{cov}(\mathcal{N}) > \omega_1$ , there is a compact space  $S$  with  $t_0(S) > \omega$ , such that  $t_0(L) = \omega$  for every continuous image  $L$  of  $S$  of weight  $\omega_1$ .*

Our result is based on the construction described in the following lemma.

**Lemma 5.5.** *Let  $(s_n)_n$  be a pairwise disjoint sequence in  $\mathfrak{A}^+$ . Let*

$$\mathcal{F} = \{a \in \mathfrak{A} : \lim_n \lambda(a \cap s_n) / \lambda(s_n) = 1\},$$

$$F = \{x \in \text{ult}(\mathfrak{A}) : \mathcal{F} \subseteq x\}.$$

*Then*

- (i)  $\mathcal{F}$  is a non-principal filter in  $\mathfrak{A}$ ;
- (ii)  $F$  is a closed subset of  $\text{ult}(\mathfrak{A})$  with empty interior;
- (iii) for every countable  $Y \subseteq \text{ult}(\mathfrak{A}) \setminus F$  we have  $\overline{Y} \cap F = \emptyset$ .



*Proof.* Part (i) follows by standard calculations and part (ii) is a direct consequence of (i). We shall check (iii). Let  $Y = \{y_n : n \in \omega\} \subseteq \text{ult}(\mathfrak{A}) \setminus F$ . For every  $n$  we have  $y_n \notin F$  so there is  $a_0^n \in y_n$  such that  $-a_0^n \in \mathcal{F}$ . Then we choose a decreasing sequence  $(a_k^n)_k$  such that

- (a)  $a_k^n \leq a_0^n$ ,  $a_k^n \in y_n$  for every  $k$
- (b)  $\lim_k \lambda(a_k^n) = 0$ .

The following fact can be proved by a standard diagonalization (cf. [15]).

CLAIM. There is a function  $g : \omega \rightarrow \omega$  such that writing  $a_g := \bigcup_{n \in \omega} a_{g(n)}^n$ , we have  $\widehat{a}_g \cap F = \emptyset$ .

Using Claim we get  $Y \subseteq \widehat{a}_g$  and it follows that  $\overline{Y} \cap F = \emptyset$ . □

*Proof.* (of Theorem 5.4). Let  $S$  be the Stone space of the measure algebra  $\mathfrak{A}$ . Take the set  $F \subseteq S$  from Lemma 5.5. Then condition (iii) implies that the function  $\chi_F : S \rightarrow \mathbb{R}$  is continuous on every countable subspace of  $S$ . But  $\chi_F$  is clearly not continuous because the interior of  $F$  is empty. Hence  $t_0(S) > 0$ .

Let now  $L$  be a continuous image of  $S$  such that  $w(L) \leq \omega_1$ . Then  $L$  is separable by Theorem 5.3 and  $t_0(L) = \omega$  by Lemma 5.2, so the proof is complete. □

*Remark 5.6.* We enclose some remarks concerning Theorem 5.4

- (1) Lemma 5.4 originates in Kunen [15]; see Plebanek [21] for other applications.
- (2) In fact, under  $\text{MA}(\omega_1)$  one can check that in the proof we actually get  $t_0(S) > \omega_1$ , since under  $\text{MA}(\omega_1)$  one can strengthen (iii) of Lemma 5.5 to saying that  $\overline{Y} \cap F = \emptyset$  for every  $Y \subseteq S \setminus F$  with  $|Y| \leq \omega_1$ .
- (3) The proof of 5.4 says a bit more, that  $t_0(L) = \omega$  whenever  $L$  is a continuous image of  $S$  having a  $\pi$ -base of cardinality  $\leq \omega_1$ .

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